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A characterization and equations for minimal shape-preserving projections

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Abstract

Let X denote a (real) Banach space and V an n-dimensional subspace. We denote by $\mathscr{B} = \mathscr{B}(X, V)$ the space of all bounded linear operators from X into V; let $\mathscr{P}(X, V)$ be the set of all projections in $\mathscr{B}$. For a given cone $S \subset X$, we denote by $\mathscr{P} = \mathscr{P}_S(X, V)$ the set of operators $P \in \mathscr{P}$ such that $PS \subset S$. When $\mathscr{P}_S \neq \emptyset$, we characterize those $P \in \mathscr{P}_S$ for which $\|P\|$ is minimal. This characterization is then utilized in several applications and examples.

1. Introduction

Over the last 30 years much has been written on the subject of minimal projections. Much of this work involves, in one way or another, the determination of a projection $P_{\min}$ from a Banach space X onto (finite-dimensional) subspace V such that $\|P_{\min}\|$ is minimized over all projections from X onto V. The significance of this problem is well illustrated in [1,3,7,8,12].

In the (frequent) setting in which the subspace V is finite-dimensional there is never a question of existence—such spaces are always complemented and, moreover, a projection of minimal norm (from X onto V) always exists. Indeed, as described in a characterization of minimality from [4], existence follows from the fact that the minimal projection problem is equivalent to a best-approximation problem in a $C(K)$ space (specifically the best approximation of a fixed function from a linear (approximating) function-space defined on the compact set K). We note that this characterization has been successfully employed in a variety of settings (see e.g. [2–5]).

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All of this changes, however, once we place a constraint on the projections from X onto V; in particular if require such projections to leave invariant a specified cone, existence of such operators is immediately called into question. Furthermore, such a constraint eliminates the use of the characterization in [4]—in the language of this characterization, there is no longer an obvious linear space from which to best-approximate.

If $S \subset X$ is a *cone* (a convex set, closed under nonnegative scalar multiplication) and $P : X \to V$ a projection, we say P is a *shape-preserving projection* if $PS \subset S$. In this paper, we use techniques from [10] to develop a characterization for minimal-norm shape-preserving projections. Not surprisingly, the theory of existence of shape-preserving projections plays an important part in this characterization. Indeed, existence and the minimal-norm characterization are so closely connected we include as part of this paper a portion of existence theory.

Specifics regarding the cones under consideration are contained in Section 2. We also include in this section properties of such cones; since they are infinite-dimensional it is necessary to verify certain basic properties. Section 3 describes a geometric equivalence to the existence of shape-preserving projections and Section 4 utilizes this condition to characterize minimal-norm shape-preserving projections. The final section demonstrates how the theory of the previous sections comes together to solve (nontrivial) problems in various classical settings. We note here that each application in Section 4 examines particular aspects of open questions in projection theory; as such, in addition to illustrating the theory, these examples are meant to serve as starting points for further investigations.

2. General preliminaries

Throughout this paper X will denote a real Banach space with unit ball and sphere denoted by $B(X)$ and $S(X)$, respectively. For fixed positive integer n, $V \subset X$ will always denote an n-dimensional subspace of X. For a given V and X, $\mathcal{B} = \mathcal{B}(X, V)$ will denote the set of linear operators from X into V, while $\mathcal{P} \subset \mathcal{B}$ will denote the set of all projections.

We now review some basic terminology from convex analysis. In a (real) topological vector space, a cone K is a convex set, closed under nonnegative scalar multiplication. K is *pointed* if it contains no lines. For $\phi \in K$, let $[\phi]^+ := \{\alpha\phi \mid \alpha \geqslant 0\}$. We say $[\phi]^+$ is an *extreme ray* of K if $\phi = \phi_1 + \phi_2$ implies $\phi_1, \phi_2 \in [\phi]^+$ whenever $\phi_1, \phi_2 \in K$. We let $E(K)$ denote the union of all extreme rays of K. When K is a closed, pointed cone of finite dimension we always have $K = \mathrm{co}(E(K))$ (this need not be the case when K is infinite dimensional; indeed, we note in [11] that it is possible that $E(K) = \emptyset$ despite K being closed and pointed). We say that a closed, pointed cone K of finite dimension is *simplicial* whenever the number of extreme rays of K is exactly $\dim(K)$.

Definition 2.1. Let X be a (fixed) Banach space and $V \subset X$ a (fixed) n-dimensional subspace. Let $S \subset X$ denote a closed cone. We say that $x \in X$ *has shape (in the sense of S)* whenever $x \in S$. If $P \in \mathcal{P}$ and $PS \subset S$ then we say P is a *shape-preserving projection*; we denote the set of all such projections by $\mathcal{P}_S = \mathcal{P}_S(X, V)$. For a given cone S, define $S^* = \{\phi \in X^* \mid \langle x, \phi\rangle \geqslant 0 \ \forall x \in S\}$. We will refer to S^* as the *dual cone* of S.

The cone S^* will play an important role in our characterization of the minimal norm element from $\mathcal{P}_S(X, V)$. We will assume throughout this paper that S^* is pointed and contains at least $n + 1$ linearly independent elements. Note that S^* is a weak*-closed cone which is "dual" to S in the following sense.

Lemma 2.1. *Let $x \in X$. If $\langle x, \phi \rangle \geqslant 0$ for all $\phi \in S^*$ then $x \in S$.*

Proof. We prove the contrapositive; suppose $x \in X$ such that $x \notin S$. Then, since S is closed and convex, there exists a separating functional $\phi \in X^*$ and $\alpha \in \mathbb{R}$ such that $\langle x, \phi \rangle < \alpha$ and

$$\langle s, \phi \rangle > \alpha \ \forall s \in S. \tag{1}$$

Note that we must have $\alpha < 0$ because $0 \in S$. In fact, for every $s \in S$ we claim

$$\langle s, \phi \rangle \geqslant 0 > \alpha. \tag{2}$$

To check this, suppose there exists $s_0 \in S$ such that $\langle s_0, \phi \rangle = \beta < 0$; this would imply

$$\left\langle \frac{\alpha}{\beta} s_0, \phi \right\rangle = \alpha$$

while $\frac{\alpha}{\beta} s_0 \in S$. And this is in contradiction to (1). The validity of (2) implies that $\phi \in S^*$ and this completes the proof. $\square$

Lemma 2.2. *Let $P \in \mathcal{B}(X, V)$. Then $PS \subset S \iff P^*S^* \subset S^*$.*

Proof. The proof is an immediate consequence of the duality equation $\langle Pf, u \rangle = \langle f, P^*u \rangle$ and Lemma 2.1. $\square$

Before characterizing minimal-norm elements of $\mathcal{P}_S(X, V)$, we must first ensure that $\mathcal{P}_S(X, V) \neq \emptyset$; we employ S^* for this task. We begin by looking for (convenient) subsets of S^* which can be used to 'recover' the entire cone. The following proposition describes one possible subset.

Proposition 2.1. *Every nonzero element of S^* is contained in the convex hull of ∂S^*.*

Proof. Let $x \in S^* \setminus \{0\}$. If $x \in \partial S^*$ then x is interior to a line segment joining 0 and a positive scalar multiple of x and thus we are finished. Suppose x belongs to the interior of S^* and let $e \in \partial S^*$. For $\alpha \in \mathbb{R}^+$, let $L_e(\alpha) \subset X^*$ denote the half-line beginning at e and passing through αx. We claim that for some $\alpha > 0$ we have $(L_e(\alpha) \cap \partial S^*) \setminus \{e\} \neq \emptyset$. Suppose this is not true; then for every $\alpha > 0$ the half-line $L_e(\alpha)$ is entirely contained in S^*. And since S^* is closed, this implies that $L_e(0)$ is a half-line entirely contained in S^*. But this contradicts that fact that S^* is pointed and our claim is established. Let $\alpha_x > 0$ be as in the above claim and let $y_x := (L_e(\alpha_x) \cap \partial S^*) \setminus \{e\}$. Thus

$$\alpha_x x = \lambda e + (1 - \lambda) y_x \tag{3}$$

for some $0 < \lambda < 1$. Dividing both sides of 3 by α_x we that x is on the line segment joining boundary elements $\frac{1}{\alpha_x} e$ and $\frac{1}{\alpha_x} y_x$. $\square$

In general, however, ∂S^* is difficult to describe and thus is of limited utility in representing elements of S^*. Another natural subset to consider is $E(S^*)$, the set of all extreme rays of S^*. The following definition provides a description of those S^* for which $E(S^*)$ provides the right recovery set. In the context of our current considerations, we say a finite (possibly) signed measure μ with support $E \subset X^*$ is a *generalized representing measure* for $\phi \in X^*$ if $\langle x, \phi \rangle = \int_E \langle s, x \rangle \, du(s)$ for all $x \in X$. A nonnegative measure μ satisfying this equality is simply a *representing measure*.

Definition 2.2. We say that S^* (the pointed dual cone of S) is *simplicial* if S^* can be recovered from its extreme rays (i.e., $S^* = \overline{\text{co}}(E(S^*)))$ and the set of extreme rays form an independent set (independent in the sense that, any generalized representing measure supported on $E(S^*)$ for $\phi \in S^*$ must be a representing measure). A pointed, closed cone of finite dimension k is simplicial if there exist exactly k (extreme) rays of the cone whose convex hull is the entire cone.

Unless otherwise noted, S^* is assumed to be simplicial. Equipped with this assumption, the following theorem provides us with an easily applied test to determine if $\mathcal{P}_S(X, V) \neq \emptyset$.

Theorem 2.1 (*see Mupasiri and Prophet [10]*). *$\mathcal{P}_S(X, V) \neq \emptyset$ if and only if the cone $S^*_{|V}$ is simplicial.*

Note 2.1. Suppose $S^*_{|V}$ is k-dimensional where $1 \leqslant k \leqslant n$. Choose a basis for V, $v_1, \ldots, v_n$ such that, for $i = 1, \ldots, n-k$, $\langle v_i, u\rangle = 0$ for all $u \in S^*$ and, for $i = n-k+1, \ldots, n$, $v_i \in S$. With this basis, any operator $P : X \to V$ can be written in the form $P = u_1 \otimes v_1 + \cdots + u_n \otimes v_n$ for some choice of u_i's $\in X^*$, where $Pf = \langle f, u_1\rangle v_1 + \cdots + \langle f, u_n\rangle v_n$ (for convenience, we often write $\boldsymbol{u} := (u_1, \ldots, u_n) \in (X^*)^n$, $\boldsymbol{v} := (v_1, \ldots, v_n)^T \in V^n$ and $Pf = \langle f, \boldsymbol{u}\rangle \boldsymbol{v}$). Thus we note that $P : X \to V$ is shape-preserving if and only if $P_1 : X \to V_1$ is shape-preserving where $V_1 := [v_{n-k+1}, \ldots, v_n]$ and $P_1 = u_{n-k+1} \otimes v_{n-k+1} + \cdots + u_n \otimes v_n$.

Corollary 2.1. *Suppose $P \in \mathcal{P}_S(X, V)$. If $S^*_{|V}$ is k-dimensional then there exists a basis $\boldsymbol{v} = (v_1, \ldots, v_n)^T$ for V such that whenever $P = \boldsymbol{u} \otimes \boldsymbol{v} \in \mathcal{P}_S(X, V)$, where $\boldsymbol{u} = (u_1, \ldots, u_n) \in (X^*)^n$, we have, for $i = n-k+1, \ldots, n$, $u_i \in S^*$. Moreover, each such u_i restricts to a distinct extreme ray of $S^*_{|V}$.*

Proof. $\mathcal{P}_S(X, V) \neq \emptyset$ implies that $S^*_{|V}$ has exactly k edges and is expressible as

$$S^*_{|V} = \text{cone}(u_{n-k+1}{}_{|V}, \ldots, u_n{}_{|V})$$

for some $(u_{n-k+1}, \ldots, u_n) =: \boldsymbol{u} \in (S^*)^n$. Choose a basis $\{v_1, \ldots, v_n\}$ for V as in Note 2.1 and define $P := \boldsymbol{u} \otimes M^{-1}\boldsymbol{v} : X \to V_1$ where $V_1 := [v_{n-k+1}, \ldots, v_n]$, $\boldsymbol{v} = (v_{n-k+1}, \ldots, v_n)^T$ and $M = \langle \boldsymbol{v}, \boldsymbol{u}\rangle = (\langle v_i, u_j\rangle)$. Obviously P is a projection onto V_1; the fact that P is shape-preserving follows from Lemma 3 in [10]. Let $Q := \boldsymbol{\phi} \otimes M^{-1}\boldsymbol{v}$ be an arbitrary projection onto V_1 preserving S. This implies that $\boldsymbol{\phi} \in (S^*)^n$, since $\boldsymbol{\phi} = Q^*\boldsymbol{u}$ and $Q^*\boldsymbol{u} \in (S^*)^n$ by Lemma 2.2. Furthermore, $\boldsymbol{\phi}_{|V} = \boldsymbol{u}_{|V}$ since $I_n = \langle M^{-1}\boldsymbol{v}, \boldsymbol{\phi}\rangle = \langle M^{-1}\boldsymbol{v}, \boldsymbol{u}\rangle$. Therefore, the only way to construct a projection onto V_1 preserving S is to "extend" the extreme rays of $S^*_{|V}$ from V to X. This observation, taken together with the concluding remark of Note 2.1, completes the proof. $\square$

Corollary 2.2. *Suppose $\mathcal{P}_S(X, V) \neq \emptyset$ and $S^*_{|V}$ is n-dimensional. If elements of S^* have restricted uniquely to $E(S^*_{|V})$ then $\mathcal{P}_S(X, V)$ contains a unique element.*

Proof. This is an immediate consequence of Corollary 2.1. $\square$

3. Characterization

In order to determine minimal norm elements of $\mathcal{P}_S(X, V)$, we will cast the problem in a 'continuous function on a compact set' setting, analogous to the approach taken in Theorem 1 of [4].

Definition 3.1. $(x, y) \in S(X^{**}) \times S(X^*)$ will be called an extremal pair for $P \in \mathcal{P}_S(X, V)$ if $\langle P^{**}x, y\rangle = \|P\|$, where $P^{**} : X^{**} \to V$ is the second adjoint extension of P to X^{**}.

Notation. Let $\mathcal{E}(P)$ be the set of all extremal pairs for P. To each $(x, y) \in \mathcal{E}(P)$ associate the rank one operator $y \otimes x$ from X to X^{**} given by $(y \otimes x)(z) = \langle z, y\rangle x$ for $z \in X$.

Theorem 3.1 (*Characterization of minimal P in $\mathcal{P}_S$*). *Let $\mathcal{P}_S(X, V)$ be nonempty. Suppose that $S^*_{|V}$ is k-dimensional, $1 \leqslant k < n$ and S^* restricts uniquely to $E(S^*_{|V})$. Then $P \in \mathcal{P}_S(X, V)$ has minimal norm in $\mathcal{P}_S(X, V)$ if and only if the closed convex hull of $\{y \otimes x\}_{(x,y)\in\mathcal{E}(P)}$ contains an operator carrying V_0 into V, where $V_0 := V \cap \left((S^*)^{\perp}\right)$. If $k = n$ then $|\mathcal{P}_S(X, V)| = 1$.*

Proof. In the case $k = n$ $|\mathcal{P}_S(X, V)| = 1$ by Corollary 2.2; thus we now assume $k < n$. Fix basis $v_1, v_2, \ldots, v_n$ for V such that $V_0 := [v_1, \ldots, v_{n-k}] \in (S^*)^{\perp}$ and $V_1 := [v_{n-k+1}, \ldots, v_n]$. Let $P_0 = \sum_{i=1}^n u_i \otimes v_i \in \mathcal{P}_S(X, V)$ where $u_i \in X^*$ for $i = 1, \ldots, n$. Then, from the conclusion of Note 2.1, we see that $P_1 := \sum_{i=n-k+1}^n u_i \otimes v_i$ is a projection (on X) preserving S onto V_1. Moreover, Corollary 2.2 (replacing n with k) indicates that P_1 is the unique projection preserving S onto V_1. Consequently, the functionals $u_{n-k+1}, \ldots, u_n$ appearing in the definition of P_0 are unique (among all possible choices of functionals defining a shape-preserving projection onto V with respect to basis $v_1, \ldots, v_n$). Therefore, the problem of finding a minimal-norm element from $\mathcal{P}_S(X, V)$ is equivalent to best approximating, in the operator norm, the fixed operator $P_0 \in \mathcal{P}_S(X, V)$ from the space of operators $\mathcal{D} = \left\{\sum_{i=1}^{n-k} \varepsilon_i \otimes v_i \mid \varepsilon_i \in (S^*)^{\perp}\right\} = \mathrm{sp}\{\delta \otimes v : \delta \in (S^*)^{\perp},\ v \in V_0\}$. Let $K = B(X^{**}) \times B(X^*)$ endowed with the product topology, where $B(\cdot^*)$ denotes the unit ball with its weak* topology. Associate with any operator $Q \in \mathcal{B}$ the bilinear form $\widehat{Q} \in C(K)$ via $\widehat{Q}(x, y) = \langle Q^{**}x, y\rangle$, and let $\widehat{\mathcal{D}} = \{\widehat{\Delta} : \Delta \in \mathcal{D}\}$. Then, making use of standard duality theory for $C(K)$, K compact (see e.g., [13], Theorem 1.1 (p. 18) and Theorem 1.3 (p. 29)), we have that $\widehat{P} = \widehat{P_0} - \widehat{\Delta_0}$ is of minimal norm if and only if there exists a finite, nonzero (total mass one) signed measure $\hat{\mu}$ supported on the critical set

$$\mathcal{C}(\widehat{P}) = \{(x, y) \in S(X^{**}) \times S(X^*) : |\widehat{P}(x, y)| = \|\widehat{P}\|_\infty\}$$

such that $\mathrm{sgn}\,\hat{\mu}\{(x, y)\} = \mathrm{sgn}\,\widehat{P}(x, y)$ and $\hat{\mu} \in \widehat{\mathcal{D}}^{\perp}$, i.e.,

$$0 = \int_{\mathcal{C}(\widehat{P})} \widehat{\Delta}\, d\hat{\mu} \quad \text{for all } \widehat{\Delta} \in \widehat{\mathcal{D}}.$$

But now, since any $\widehat{Q} \in \{\widehat{P}\} \cup \widehat{\mathcal{D}}$ is a bilinear function, we can replace the signed measure $\hat{\mu}$, supported on $\mathcal{C}(\widehat{P})$, by a positive measure μ supported on $\mathcal{E}(P) \subset \mathcal{C}(\widehat{P})$ by noting that

$$\mathcal{C}(\widehat{P}) = \{(x, \pm y) : (x, y) \in \mathcal{E}(P)\}$$

and setting

$$\mu\{(x, y)\} = |\hat{\mu}|\{(x, y), (x, -y)\}.$$

For then $\mathrm{sgn}\,\mu\{(x, y)\} = \mathrm{sgn}\,\widehat{P}(x, y) = 1$, for $(x, y) \in \mathcal{E}(P)$ and

$$0 = \int_{\mathcal{E}(P)} \widehat{\Delta}\, d\mu \quad \text{for all } \Delta \in \mathcal{D},$$

since

$$\int_{\mathcal{C}(\widehat{P})} \widehat{\Delta}\, d\hat{\mu} = \int_{\substack{(x,y)\in\mathcal{E}(P)\\ k\in\{0,1\}}} \widehat{\Delta}(x, (-1)^k y)\, d\hat{\mu}(x, (-1)^k y)$$
$$= \int_{\substack{(x,y)\in\mathcal{E}(P)\\ k\in\{0,1\}}} (-1)^k \widehat{\Delta}(x, y)\, (-1)^k\, d|\hat{\mu}|(x, (-1)^k y)$$
$$= \int_{\mathcal{E}(P)} \widehat{\Delta}\, d\mu.$$

Hence,

$$0 = \int_{\mathcal{E}(P)} \widehat{\Delta}\, d\mu = \int_{\mathcal{E}(P)} \langle \Delta^{**} x, y\rangle\, d\mu(x, y)$$
$$= \int_{\mathcal{E}(P)} \langle x, \delta\rangle \langle v, y\rangle\, d\mu(x, y)$$
$$= \left\langle \int_{\mathcal{E}(P)} \langle v, y\rangle x\, d\mu(x, y),\quad \delta \right\rangle$$

for all $\Delta = \delta \otimes v$ ($\delta \in (S^*)^{\perp}$, $v \in V_0$), where, for $z \in X$, $\int_{\mathcal{E}(P)} \langle z, y\rangle x\, d\mu(x, y)$ is the element $w \in X^{**}$ defined by $\langle x^*, w\rangle = \int_{\mathcal{E}(P)} \langle z, y\rangle \langle x^*, x\rangle\, d\mu(x, y)$ for all $x^* \in X^*$. P is minimal, therefore, if and only if $\int_{\mathcal{E}(P)} \langle v, y\rangle x\, d\mu(x, y) \in (V_n^{\perp})^{\perp} = V_n$, i.e., if and only if there exists an operator (from X into X^{**})

$$E_P = \int_{\mathcal{E}(P)} y \otimes x\, d\mu(x, y) : \quad V_0 \to V_n. \qquad \square$$

The following two corollaries describe results that can be obtained without assuming that S^* is simplicial (see Definition 2.2). Both corollaries will be utilized in Section 4.2 where we consider an example in which S^* possesses no extreme rays.

Corollary 3.1. *Let S^* be the dual cone of cone $S \subset X$. Let V be an n-dimensional subspace of X. If $S^*_{|V}$ is simplicial then $\mathcal{P}_S(X, V)$ is nonempty.*

Proof. Consider the proof of Corollary 2.1; once it is established that $S^*_{|V}$ is simplicial (which happens in the first sentence of the proof), a shape-preserving projection P is immediately constructed. $\square$

Corollary 3.2. *Let S^* be the dual cone of cone $S \subset X$. Let V be an n-dimensional subspace of X. Assume $S^*_{|V}$ is simplicial with dimension k where $1 \leqslant k < n$. Fix basis $v_1, v_2, \dots, v_n$ for V such that $V_0 := [v_1, \dots, v_{n-k}] \in (S^*)^{\perp}$ and $V_1 := [v_{n-k+1}, \dots, v_n]$. Fix*

$$P_0 = u_1 \otimes v_1 + \cdots u_{n-k} \otimes v_{n-k} + u_{n-k+1} \otimes v_{n-k+1} + \cdots u_n \otimes v_n \in \mathcal{P}_S$$

and denote by $\mathcal{P}_S^0$ the set of all $P = \sum_{i=1}^{n} w_i \otimes v_i \in \mathcal{P}_S$ such that

$$w_j = u_j \quad \text{for } j = n - k + 1, \dots, n.$$

Then $P \in \mathcal{P}_S^0$ is minimal if and only if the closed convex hull of $\{y \otimes x\}_{(x,y)\in\mathcal{E}(P)}$ contains an operator carrying V_0 into V.

Proof. Analogous proof of Theorem 3.1, the collection $\mathcal{P}_S^0$ can be expressed as $P_0 - \mathcal{D}$ where

$$\mathcal{D} = \left\{ \sum_{i=1}^{n-k} \varepsilon_i \otimes v_i \mid \varepsilon_i \in (S^*)^\perp \right\} = \mathrm{sp}\{\delta \otimes v : \ \delta \in (S^*)^\perp, \ v \in V_0\}.$$

Thus the problem of finding a minimal norm element from $\mathcal{P}_S^0$ is equivalent to best approximating (in the operator norm) P_0 from $\mathcal{D}$. From here the proof proceeds as in that of Theorem 3.1. $\square$

4. Applications

We now apply the above minimization theory in various classical settings. As is clear from the development, minimal shape-preserving projection theory is closely connected with existence theory. Existence of shape-preserving projections relies on the relationships between three 'players': the overspace X, subspace $V \subset X$ and the shape $S \subset X$ to be preserved (or equivalently $S^* \subset X^*$). As the following examples illustrate, relatively small changes in the triple (X, V, S) greatly impact $\mathcal{P}_S(X, V)$.

Example 4.1. Let $X = C^2[0, 1]$ and $V = \Pi_4$—the space of 4th-degree algebraic polynomials considered as a subspace of $C^2[0, 1]$. Let $S \subset X$ denote the cone of convex functions. In this case S^* is simplicial with the set of extreme rays $E(S^*) = \{[\delta_t'']^+\}_{t \in [0,1]}$, where $\delta_t'' \in X^*$ denotes 2nd-derivative evaluation at t. As verified in [5], this combination of X, V and S forces $\mathcal{P}_S(X, V) = \emptyset$. However, by changing to $V = \Pi_3$, we find that $S^*_{|V}$ is simplicial and thus $\mathcal{P}_S(X, V) \neq \emptyset$.

Example 4.2. Let $X = C[0, 1]$ and $V = \Pi_2$. Let $S \subset X$ denote the cone of nonnegative functions. In this case S^* is simplicial with the set of extreme rays $E(S^*) = \{[\delta_t]^+\}_{t \in [0,1]}$ where $\delta_t \in X^*$ denotes point-evaluation at t. It is immediately clear that $S^*_{|V}$ fails to be simplicial; indeed each extreme ray of S^* restricts to a unique extreme ray of $S^*_{|V}$. Thus $\mathcal{P}_S(X, V) = \emptyset$. Now consider the following small variation: let $\phi \in X^*$ denote any functional such that

$$\langle 1, \phi \rangle = \alpha, \quad \langle x, \phi \rangle = \beta \quad \text{and} \quad \langle x^2, \phi \rangle = 0,$$

where $\beta/\alpha \geqslant 1/2$. Let $S_1^* = \overline{\mathrm{co}}(S^* \cup [\phi]^+)$ and define $S_1 := \{x \in X \mid \langle x, u \rangle \geqslant 0 \ \forall u \in S_1^*\}$. It is easy to verify that $(S_1)_{|V}$ is simplicial and therefore the S_1 shape can be preserved by a projection onto V; i.e., $\mathcal{P}_{S_1}(X, V) \neq \emptyset$.

Example 4.3. Let $X = C[0, 1]$ and $V = \Pi_2$. Let $S \subset X$ denote the cone of nondecreasing functions. In Lemma 4 of [10], it is demonstrated that, regardless of whether S^* is simplicial, the cone $S_{|V}$ must be closed in order for $\mathcal{P}_S(X, V) \neq \emptyset$. We now show that this cone fails to be closed. Consider $S^*_{|V}$: since every element of this cone vanishes on the identically 1 function, we can regard $S^*_{|V}$ as a subset of $\mathbb{R}^2$ by associating each $\phi_{|V} \in S^*_{|V}$ with the 2-tuple $\big(\langle x, \phi \rangle, \langle x^2, \phi \rangle\big)$. We claim that the ray determined by $e_1 := (1, 0)$ does not belong to the cone. Suppose, to the contrary, that there exists $\phi \in S^*$ such that $\phi_{|V} = (1, 0)^T$. Let m be an arbitrary positive integer and consider the function $F(t) := mt^2 - G(t)$, where $G(t)$ is any C^1 function such that $0 \leqslant G'(t) \leqslant 2mt$ for all $t \in [0, 1]$. F is monotone so $\langle F, \phi \rangle \geqslant 0$; but G is also monotone and ϕ vanishes on t^2. The only possibility then is that ϕ vanishes on G. However, vanishing on all such G leads quickly to the conclusion that ϕ is unbounded. Therefore the ray determined by e_1 does not belong to the cone and, moreover, the cone is not closed. Therefore $\mathcal{P}_S(X, V) = \emptyset$. However, if we change to

$X = C^1[0,1]$ (but keep $V = \Pi_2$ and $S \subset C^1[0,1]$ as the cone of nondecreasing functions) we find $S^*_{|V}$ to be simplicial and thus $\mathcal{P}_S(X, V) \neq \emptyset$.

4.1. $C^m[a, b]$

For fixed positive integer m let X denote the mth continuously differentiable functions, $C^m[a, b]$, normed by $\|f\| := \max_{i=0\ldots m}\{\|f^{(i)}\|_\infty\}$. In this setting, note that $\delta_t^{(k)}$, kth derivative evaluation at t belongs to sphere of X^* whenever $0 \leqslant k \leqslant m$ and $t \in [a, b]$. Moreover, for fixed k, the cone $S^* := \overline{\text{cone}}(\{\delta_t^{(k)}\}_{t\in[a,b]})$ is simplicial, as per Definition 2.2, since $E_0 = \{\delta_t^{(k)}\}_{t\in[a,b]}$. We refer to the corresponding cone $S \subset X$ as the set (or cone) of k-convex functions. In [5], with $V := \Pi_m$ (the mth degree algebraic polynomials) it is shown that $\mathcal{P}_S(X, V) = \emptyset$ if and only if $k < m - 1$. For example, there is no monotonicity-preserving (1-convex preserving) projection from X onto the Π_3. In the $k = m - 1$ case, the cone $S^*_{|V}$ is simplicial—2-dimensional and closed. This implies that, of the $m + 1$ functionals needed to define a minimal element from $\mathcal{P}_S(X, V)$, 2 of these functionals will be given by the extreme rays of $S^*{}_{|V}$ and the remaining $m - 1$ must be determined. That is, the dimension of V_0 (as in Theorem 3.1) is $m - 1$. Theorem 4.2 in [5] identifies a minimal element of $\mathcal{P}_S(X, V)$ (with norm 3/2 for every k) by constructing an E_P operator mapping V_0 into V.

The results in [5] are exclusively related to the case $k = m - 1$, where the cone $S^*_{k\,|V}$ is 2-dimensional (and closed) and hence automatically simplicial (the case $k = m$ is trivial in the sense that $\dim(S^*_{k\,|V}) = 1$). One direction of generalization (that results in higher-dimensional cones $S^*_{k\,|V}$) is to seek preservation of “multi-convex” shapes, in the following sense. Using the notation of [9], let $\boldsymbol{\sigma} = \{\sigma_i\}_{i=0,\ldots,m}$ be an $(m + 1)$-tuple with $\sigma_i \in \{-1, 0, 1\}$ and define $S_{\boldsymbol{\sigma}} := \{f \in X \mid \sigma_i f^{(i)} \geqslant 0,\ i = 0, \ldots, m\}$. With $V = \Pi_m$ fixed, one may look for all $\boldsymbol{\sigma}$ such that $\mathcal{P}_{S_{\boldsymbol{\sigma}}} \neq \emptyset$. The following example illustrates the case of $m = 3$ and $\sigma = (0, 1, 1, 0)$, in which the associated cone $S^*_{|V}$ is 3-dimensional and simplicial.

Example 4.4. Let $X = C^{(2)}[0, 1]$ and let $V = \Pi_3$ with fixed basis (vector) $\boldsymbol{v} = [1, x, x^2, x^3]^T$. Let

$$S^* := \overline{\text{cone}}\left(\{\delta'_t\}_{t\in[0,1]} \cup \{\delta''_t\}_{t\in[0,1]}\right).$$

Thus $S \subset X$ is the cone of convex, monotone-increasing functions. Note that for $t \in [0, 1]$ we have

$$(\delta''_t)_{|V} = (1 - t)(\delta''_0)_{|V} + t(\delta''_1)_{|V}$$

and

$$(\delta'_t)_{|V} = (\delta'_t)_{|V} + \left(t - \frac{t^2}{2}\right)(\delta''_0)_{|V} + \frac{t^2}{2}(\delta''_1)_{|V}.$$

Thus $S^*_{|V}$ is a (3-dimensional) simplicial cone with extreme rays $[(\delta'_0)_{|V}]^+$, $[(\delta''_0)_{|V}]^+$ and $[(\delta''_1)_{|V}]^+$. By Theorem 3.1, $\mathcal{P}_S(X, V) \neq \emptyset$. Moreover, by Corollary 2.1, every $P = \sum_{i=1}^4 u_i \otimes x^{i-1} = \boldsymbol{u} \otimes \boldsymbol{v} \in \mathcal{P}_S(X, V)$, must have $u_2 = \delta'_0$, $u_3 = \frac{1}{2}\delta''_0$ and $u_4 = \frac{1}{6}(\delta''_1 - \delta''_0)$. Thus

$$P = u_1 \otimes 1 + \delta'_0 \otimes x + \tfrac{1}{2}\delta''_0 \otimes x^2 + \tfrac{1}{6}(\delta''_1 - \delta''_0) \otimes x^3 \tag{4}$$

for some $u_1 \in X^*$. We now proceed with the construction of u_1 so that P is a minimal norm element of $\mathcal{P}_S(X, V)$. In this process we will demonstrate how the theory of Theorem 3.1 guides the construction. We begin by identifying extremal pairs for P. Recall the $P \in \mathcal{P}_S$ is minimal if and only there is a convex combination of extremal pairs (denoted by E_P) mapping V_0 into V, where V_0 is the 1-dimensional space spanned by $v(t) = 1$. Consider possible extremal pairs of the form (F_0, δ_t), (F_1, δ_t') and (F_2, δ_t''), where $F_i \in S(X^{**})$ for $i = 0, 1, 2$. From the above forms of P, we see for $f \in S(X)$, that

$$|(Pf)''(x)| = |(1-x)f''(0) + f''(1)| \leqslant 1$$

and

$$|(Pf)'(x)| = \left|\left(-\frac{1}{2}x^2 + x\right) f''(0) + \frac{x^2}{2} f''(1) + f'(0)\right| \leqslant 2. \tag{5}$$

From this we conclude that, while no extremal pair of the form (F_2, δ_t'') can exist, (we assume the norm of the minimal shape-preserving projection will exceed 1) we may have an extremal of the form (F_1, δ_t'). Before attempting to construct such a pair, let us briefly consider elements in X^{**}. Let x_n be a sequence of functions in $S(X)$ such that the set

$$M = \{f \in X^* \mid \lim_{n\to\infty} \langle x_n, f\rangle \ \text{ exists}\}$$

is nonempty. M is a subspace of X^*. Define on M the linear functional $F : M \to \mathbf{R}$ by

$$\langle f, F\rangle = \lim_{n\to\infty} \langle x_n, f\rangle.$$

Note $\|F\| \leqslant 1$. By the Hahn–Banach extension theorem, extend F to all X^* and thus $F \in S(X^{**})$. Of course, we do not know the representation of F off M.

With this construction in mind, we now define a sequence $\{x_n\}$ in X (we choose the following representation due to its utility); let

$$x_n(t) := \int_0^t z_n(x)\, dx,$$

where

$$z_n(x) := \begin{cases} -\frac{nx^2}{2} + x + \frac{2n^2-3n-4}{2n^2}, & 0 \leqslant x < 1/n, \\ -\frac{n^2-n-2}{n^2-2n+1}x^2 + \frac{2(n^2-n-2)}{n(n^2-2n+1)}x + \frac{(n-2)(n^2-n-2)}{n(n^2-2n+1)}, & 1/n \leqslant x < 1/2, \\ \frac{2n^3-5n^2+n+4}{n(n^2-2n+1)}x^2 - \frac{3n^3-8n^2+8+n}{n(n^2-2n+1)}x + \frac{7n^3-18n^2-n+20}{4n(n^2-2n+1)}, & x \geqslant 1/2. \end{cases}$$

For $n \geqslant 3$, straightforward calculations verify the following important properties of the functions $x_n(t)$:

$$\|x_n\| \in S(X), \quad x_n'(0) = \frac{(n-2)^2(n+1)}{n(n-1)^2}, \quad x_n''(0) = x_n''(1) = 1. \tag{6}$$

Now note that the subspace $M = \{f \in X^* \mid \lim_{n\to\infty}\langle x_n, f\rangle \text{ exists}\}$ contains all second derivative point-evaluations since

$$\lim_{n\to\infty}\langle x_n, \delta_t''\rangle = \begin{cases} 1, & t = 0, \\ -2t, & 0 < t < 1/2, \\ 4t-3, & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Thus we may associate $\{x_n\}$ with a functional $F_1 \in S(X^{**})$ and consider the pair (F_1, δ_1') as acting on P. Using the continuity of P^{**} and Eqs. (5) and (6) we have

$$\begin{aligned}\langle P^{**}F_1, \delta_1'\rangle = \left\langle \lim_{n\to\infty} Px_n, \delta_1'\right\rangle &= \lim_{n\to\infty}\langle Px_n, \delta_1'\rangle \\ &= \lim_{n\to\infty}\left(\left(-\frac{1}{2}+1\right)x_n''(0) + \frac{1}{2}x_n''(1) + x_n'(0)\right) \\ &= 2.\end{aligned}$$

Furthermore, if we were to define $E_P := F_1 \otimes \delta_1'$ then we would immediately obtain $E_P : V_0 \to V$ since $E_P(1) = 0$. Thus, recalling the form of P given in (4), we see that if we can choose $u_1 \in X^*$ such that, for all $F_0 \in S(X^{**})$ and all $t \in [0, 1]$, (F_0, δ_t) fails to be an extremal pair for P (i.e., if $|\langle P^{**}F_0, \delta_t\rangle| < 2$) then indeed (F_1, δ_1') will be extremal for such a P. Moreover, we will know $\|P\|$ minimal. To this end, consider

$$u_1 := \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 - \frac{1}{2}\delta_0' - \frac{1}{6}\delta_0'' - \frac{1}{12}\delta_1''.$$

For this choice of u_1 we have, for each $f \in S(X)$,

$$\begin{aligned}(Pf)(x) &= \frac{1}{2}f(0) + \frac{1}{2}f(1) + \left(x - \frac{1}{2}\right)f'(0) \\ &\quad + \left(-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{6}\right)f''(0) + \left(\frac{1}{6}x^3 - \frac{1}{12}\right)f''(1). \end{aligned} \tag{7}$$

From (7) it follows that

$$\begin{aligned}|Pf(x)| &\leqslant 1 + \left|x - \frac{1}{2}\right| + \left|-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{6}\right| + \left|\frac{1}{6}x^3 - \frac{1}{12}\right| \\ &\leqslant 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} \\ &< 2.\end{aligned}$$

Thus, a minimal norm element of $\mathcal{P}_S(X, V)$ is given by

$$\begin{aligned}P &= \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 - \frac{1}{2}\delta_0' - \frac{1}{6}\delta_0'' - \frac{1}{12}\delta_1''\right) \otimes 1 + \delta_0' \otimes x \\ &\quad + \frac{1}{2}\delta_0'' \otimes x^2 + \frac{1}{6}(\delta_1'' - \delta_0'') \otimes x^3\end{aligned}$$

with $\|P\| = 2$.

4.2. $C[a, b]$

There are natural 'shapes' in the Banach space setting $X = C[a, b]$ (equipped with the supremum norm) that one may look to preserve using a projection. Among these is the cone of monotone (increasing) functions, defined in the following way. Let $S^* \subset X^*$ denote the weak-* closure of the cone generated by all (forward) differences; i.e.,

$$S^* := \overline{\text{cone}}\left(\{\delta_{t_2} - \delta_{t_1} \mid a \leqslant t_1 < t_2 \leqslant b\}\right).$$

Clearly the induced cone $S \subset X$ contains exactly the monotone functions. It is somewhat surprising that this shape is particularly difficult to preserve onto finite-dimensional subspaces. A leading cause of this is the 'flatness' of the S^* cone, as now described.

Lemma 4.1 (*see Mupasiri and Prophet [11]*). *The weak* closed, pointed cone S^* has no extreme rays.*

The unusual structure of this S^* is revealed by combining Lemmas 2.1 and 4.1: on one hand, we see that every element of S^* is on a line segment joining 2 boundary points; on the other hand, we see that no boundary point belongs to an extreme ray—every boundary point is on a line segment joining 2 distinct boundary points.

We say that subspace $V \subset X$ is *monotonically complemented* (in X) if there exists a monotonicity preserving projection from X onto V. The following describes a large class of subspaces that fail to be monotonically complemented.

Corollary 4.1 (*see Mupasiri and Prophet [11]*). *Let V be a finite-dimensional subspace of $C[0, 1]$. If V contains the functions x^k and x^j for any two distinct positive numbers k and j then V is not monotonically complemented.*

The above demonstrates, for example, that for $n \geqslant 2$ the subspace of nth-degree algebraic polynomials Π_n is not monotonically complemented in $C[0, 1]$. In the following example, we identify a sequence of 3-dimensional monotonically complemented subspaces V_k that 'converge' to Π_2. For each k, we employ the theory from Section 3 and identify a projection which is minimal among a class of monotonicity-preserving projections.

Example 4.5. Let $X = C[0, 1]$. For $k > 2$ define $V_k := [1, x, v_k(x)]$ where

$$v_k(x) := \begin{cases} 0 & \text{if } x \leqslant 1/k, \\ \frac{k}{k-2}(x - 1/k)^2 & \text{if } 1/k \leqslant x \leqslant (k-1)/k, \\ 2x - 1 & \text{if } x \geqslant (k-1)/k. \end{cases}$$

For example, with $k = 5$, we plot both x^2 (the thicker curve) and $v_k(x)$:

Let $\boldsymbol{v} = (1, x, v_k(x))^T$. Notice that $V_0 := V_k \cap (S^*)^{\perp} = [1]$. Due to the 'flatness' of v_k on the intervals $[0, 1/k]$ and $[(k-1)/k, 1]$, we find that $S^*_{|V_k}$ is a (2-dimensional) simplicial cone. The extreme rays of this cone are (nonuniquely) generated by $\Delta_{0|V_k}$ and $\Delta_{1|V_k}$ where

$$\Delta_0 = k(\delta_{1/k} - \delta_0) \quad \text{and} \quad \Delta_1 = k(\delta_1 - \delta_{(k-1)/k}).$$

Corollary 3.1 guarantees the existence of a projection $P = \boldsymbol{u} \otimes \boldsymbol{v}$ such that $P \in \mathcal{P}_S(X, V)$. Moreover, the proof of the Corollary 3.1 indicates that u_1 and u_2 must be chosen from S^* so that they agree with (specific linear combinations of) Δ_0 and Δ_1 when restricted to V_k. We now employ this criteria on V_k-restrictions in the following manner. Though S^* is not simplicial (indeed, Lemma 4.1 indicates that it has no extreme rays), we may still seek a 'constrained' minimal projection by appealing to Corollary 3.2. Specifically, by *fixing*

$$u_1 := \Delta_0 \quad \text{and} \quad u_2 := \frac{\Delta_1 - \Delta_0}{2}, \tag{8}$$

we seek the minimal monotonicity-preserving projection onto V_k with the u_1 and u_2 as in (8). To this end, consider

$$P := (\delta_0 + \varepsilon_k) \otimes 1 + u_1 \otimes x + u_2 \otimes v_k,$$

where

$$\varepsilon_k = \frac{1}{4}\left(\frac{k(k-2)}{k-1}\delta_0 - \frac{k^2}{k-1}\delta_{1/k} + \frac{k^2}{k-1}\delta_{(k-1)/k} - \frac{k(k-2)}{k-1}\delta_1\right).$$

We claim P has minimal norm (among all those projections onto V_k which preserve S and satisfy (8)). To see this we note that the norm of P can be obtained via a Lebesgue-function approach; that is, $\|P\|$ is obtained by maximizing the quantity $\sup_{f\in B(X)} |Pf(x)|$ over $x \in [0, 1]$. From the definition of v_k, we see that we can consider the 3 cases $x \in [0, 1/k]$, $x \in [1/k, (k-1)/k]$, and $x \in [(k-1)/k, 1]$. However, $v_k = 0$ on $[0, 1/k]$ (which causes $\sup_{f\in B(X)} |Pf(x)| = 1$ on this interval) and consequently we need only consider the latter two. Elementary calculus shows that, for $x \in [(k-1)/k, 1]$, we have

$$\sup_{f\in B(X)} |Pf(1)| \geqslant \sup_{f\in B(X)} |Pf(x)|$$

and, for $x \in [1/k, (k-1)/k]$ we have

$$\sup_{f\in B(X)} |Pf(1/k)| \geqslant \sup_{f\in B(X)} |Pf(x)|.$$

A direct calculation reveals

$$\begin{aligned} Pf(1) = &-\frac{-4k+4+k^2}{4(k-1)} f(0) - \frac{-k^2+2k}{4(k-1)} f(1/k) \\ &- \frac{k^2-6k+4}{4(k-1)} f((k-1)/k) - \frac{-k^2+4k-4}{4(k-1)} f(1) \end{aligned}$$

and

$$\begin{aligned} Pf(1/k) = &\frac{k^2-2k}{4(k-1)} f(0) + \frac{-k^2+4k-4}{4(k-1)} f(1/k) \\ &+ \frac{k^2}{4(k-1)} f((k-1)/k) + \frac{-k^2+2k}{4(k-1)} f(1). \end{aligned}$$

The quantity $Pf(1)$ is maximized when

$$f(0) = -1, \quad f(1/k) = 1, \quad f((k-1)/k) = -1, \quad f(1) = 1 \tag{9}$$

and $Pf(1/k)$ is maximized for

$$f(0) = 1, \quad f(1/k) = -1, \quad f((k-1)/k) = 1, \quad f(1) = -1. \tag{10}$$

Moreover, we find that these maximized quantities are equal with common value $k-1$; i.e.,

$$\sup_{f \in B(X)} |Pf(1)| = \sup_{f \in B(X)} |Pf(1/k)| = k-1.$$

This allows us to define 2 extremal pairs. Let $F, G \in B(X)$ be the piecewise linear functions such that F interpolates as in (9) and G interpolates as in (10). Then

$$(\delta_1, F) \quad \text{and} \quad (\delta_{(k-1)/k}, G)$$

are extremal pairs for P. Let $E_P = \delta_1 \otimes F + \delta_{(k-1)/k} \otimes G$; according to Corollary 3.2 we must show that $E_P : V_0 \to V_k$, where $V_0 = [1]$. But

$$E_P(1) = \langle 1, \delta_1 \rangle F + \langle 1, \delta_{(k-1)/k} \rangle G = F + G = 0 \in V_0$$

and thus P is minimal.

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